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The Generalized Incomplete Gamma Function as sum over Modified Bessel Functions of the First Kind

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ABSTRACT

We represent the Generalized Incomplete Gamma Function

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} \exp\left(-t - \frac{b}{t}\right) \mathrm{d}t, \quad \alpha \in \mathbb{R},\ x \geq 0,\ b \geq 0,$$
$$\text{but not both } x = b = 0,\ \text{if } \alpha \leq 0,$$

as a sum of Modified Bessel Functions valid for non-integer α. For integer values of α we derive the corresponding limit case. Moreover, we discuss numerical techniques to evaluate this function.

1. Introduction

In this paper we expand the Generalized Incomplete Gamma Function defined as

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} \exp\left(-t - \frac{b}{t}\right) \mathrm{d}t, \quad \alpha \in \mathbb{R},\ x \geq 0,\ b \geq 0,\ \text{but not both } x = b = 0,\ \text{if } \alpha \leq 0, \tag{1}$$

in terms of a sum over the functions I_j, the Modified Bessel Function of the First Kind, order j. Earlier, such an expansion has been given in [1, (38)] for $\alpha = 0$. Here, we specify this expansion for general real α, where values of $\alpha \in \mathbb{Z}$ need a special treatment. Our general result is Eq. (17), while for $\alpha = 0$ we found (23) and for integer values of α, $\alpha \geq 1$, we found (30). For integer values $\alpha \leq -1$, we apply the functional equation (7). Moreover, in (33) we present an alternative representation for the erfc-function in terms of such a sum over I_j, which can be cast into the form of a Neumann Expansion (34). Finally, in the last section we discuss numerical techniques to evaluate this function.

2. Generalized Incomplete Gamma Function

The Generalized Incomplete Gamma Function has numerous applications, among others for $\alpha = 0$ in the field of hydrology [2,3]. A number of well-known statistical distributions can be expressed in terms of (1). Recently, a number of papers have been appeared (see [4,1,5]); some papers pay only attention for the case $\alpha = 0$ (see [6–9]). Book [10] summarizes a number of properties such as

$$\Gamma(\alpha, 0; b) = 2b^{\alpha/2} K_\alpha\left(2\sqrt{b}\right), \tag{2}$$

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where K_α is the Modified Bessel Function of the Second Kind, order α and gives several analytical expressions for $\Gamma(\alpha, x; b)$ as infinite sums, among others

$$\Gamma(\alpha, x; b) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \Gamma(\alpha - n, x), \quad x > 0, \ \alpha \geq 0. \tag{3}$$

with $\Gamma(\alpha, x)$ the Incomplete Gamma Function

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} \exp(-t)\, dt, \quad \alpha \geq 0, \ x \geq 0, \ \text{not both } \alpha = x = 0. \tag{4}$$

For values of $\alpha = n - 1/2$, $n \geq 0$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, [4] and [10, Theorem. 2.7 and Corollary 2.10] gives a closed-form expression as a finite sum over Bessel functions with half-integers as order, $\alpha = -1/2, 1/2, 3/2, \ldots$

$$\begin{aligned}\Gamma(\alpha, x; b) = b^{\alpha/2} \Bigg[& \left\{K_\alpha\left(2\sqrt{b}\right) + \pi(-1)^{\alpha-1/2} I_\alpha\left(2\sqrt{b}\right)\right\} \operatorname{erfc}\left(\sqrt{x} + \sqrt{b/x}\right) \\ & + K_\alpha\left(2\sqrt{b}\right) \operatorname{erfc}\left(\sqrt{x} - \sqrt{b/x}\right) + 2\exp(-x - b/x) \\ & \times \sum_{j=0}^{\alpha-3/2} \left(x/\sqrt{b}\right)^{j+1/2} \left\{ I_{j+1/2}\left(2\sqrt{b}\right) K_\alpha\left(2\sqrt{b}\right) + (-1)^{\alpha+j+1/2} K_{j+1/2}\left(2\sqrt{b}\right) I_\alpha\left(2\sqrt{b}\right)\right\}\Bigg].\end{aligned} \tag{5}$$

For values of $\alpha = n$, $n \geq 0$, $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $x = \sqrt{b}$ an explicit expression has been found as a finite sum over Bessel functions with integers as order ([11] and [10, Corollary 2.11]):

$$\begin{aligned}\Gamma(n, \sqrt{b}; b) = b^{n/2} \Bigg[& K_n\left(2\sqrt{b}\right) + \exp\left(-2\sqrt{b}\right) \left\{ I_0\left(2\sqrt{b}\right) K_n\left(2\sqrt{b}\right) + (-1)^{n+1} K_0\left(2\sqrt{b}\right) I_n\left(2\sqrt{b}\right)\right\} \\ & + 2\exp\left(-2\sqrt{b}\right) \sum_{j=0}^{\alpha-2} \left\{ I_{j+1}\left(2\sqrt{b}\right) K_n\left(2\sqrt{b}\right) + (-1)^{j+n} K_{j+1}\left(2\sqrt{b}\right) I_n\left(2\sqrt{b}\right)\right\}\Bigg].\end{aligned} \tag{6}$$

To evaluate $\Gamma(\alpha, x; b)$ for $\alpha < 0$, the functional relation

$$\Gamma(\alpha, x; b) + b^\alpha \Gamma\left(-\alpha, \frac{b}{x}; b\right) = 2b^{\alpha/2} K_\alpha\left(2\sqrt{b}\right) \tag{7}$$

is useful.

3. Expansion in terms of Bessel functions

Chaudhry and Zubair [10] discuss a result by Vu Kim Tuan as [10, (2.158)]

$$\Gamma(\alpha, x; b) = \Gamma(\alpha)\,{}_0F_1(1 - \alpha; b) - \frac{x^\alpha}{\alpha} \Gamma_2\left(-\alpha, \alpha, x, \frac{b}{x}\right), \tag{8}$$

$$\text{with } {}_0F_1(\beta; x) = \sum_{n=0}^{\infty} \frac{1}{(\beta)_n n!} x^n, \tag{9}$$

$$\text{and } \Gamma_2\left(\beta, \beta', x, y\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{n-m} (\beta')_{m-n}}{m!n!} x^m y^n, \tag{10}$$

where the symbol $(\alpha)_k$ represents the Pochhammer symbol

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1) \cdots (a + k - 1), \quad a \neq 0, -1, -2, \ldots.$$

The interpretation for the symbol $(a)_{-k}$ can be given in a natural way as [12, p. 16–17]

$$(a)_{-k} = \frac{\Gamma(a - k)}{\Gamma(a)} = \frac{(-1)^k}{(1 - a)_k}, \quad k = 1, 2, 3, \ldots, \ a \neq 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{11}$$

The function Γ_2 is a generalized hypergeometric function in two variables; see [13, p. 384] or [14, p. 226, (28)].

The first term in (8) can be written as

$$\Gamma(\alpha)_0F_1(1-\alpha;b)=\Gamma(\alpha)\Gamma(1-\alpha)b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right)=\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right),\tag{12}$$

using [15, (9.6.10) & (6.1.17)]. In (8), the second term gives rise to the expressions

$$\begin{aligned}\frac{(-\alpha)_{n-m}(\alpha)_{m-n}}{\alpha}&=\frac{(-\alpha)_{n-m}(-1)^{n-m}}{\alpha(1-\alpha)_{n-m}}=-\frac{\Gamma(-\alpha)(-\alpha)_{n-m}(-1)^{n-m}}{\Gamma(1-\alpha)(1-\alpha)_{n-m}}\\&=-\frac{\Gamma(n-m-\alpha)(-1)^{n-m}}{\Gamma(n-m+1-\alpha)}=-\frac{(-1)^{n-m}}{(n-m-\alpha)},\end{aligned}\tag{13}$$

so

$$\begin{aligned}-\frac{x^\alpha}{\alpha}\Gamma_2\left(-\alpha,\alpha,x,\frac{b}{x}\right)&=-\frac{x^\alpha}{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\alpha)_{n-m}(\alpha)_{m-n}}{m!n!}x^m\left(\frac{b}{x}\right)^n\\&=x^\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n.\end{aligned}\tag{14}$$

Taking Eqs. (12) and (14) together, we find the result

$$\Gamma(\alpha,x;b)=\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right)+x^\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n.\tag{15}$$

In Eq. (15) there is a double sum $\sum\sum_{m,n=0}^{\infty}a_{m,n}$ involved. It is possible to handle this sum by performing summation along the main diagonal ($n=m$), below ($n>m$), and above ($m>n$) the main diagonal. We make respectively the substitutions $k=n=m, j=n-m$ and $j'=m-n$ with the following result

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n\\&=\sum_{k=0}^{\infty}\frac{1}{(-\alpha)k!k!}(-b)^k+\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(j-\alpha)m!(j+m)!}(-1)^{j+2m}x^{-j}b^{j+m}\\&\quad-\sum_{j'=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(j'+\alpha)n!(j'+n)!}(-1)^{j'+2n}x^{j'}b^n\\&=-\frac{1}{\alpha}I_0\left(2\sqrt{b}\right)+\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!(j+m)!}\left(\frac{(-x)^{-j}b^{j+m}}{(j-\alpha)}-\frac{(-x)^jb^m}{(j+\alpha)}\right)\\&=-\frac{1}{\alpha}I_0\left(2\sqrt{b}\right)+\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{b^m}{m!(j+m)!}\left(\frac{(-b/x)^j}{(j-\alpha)}-\frac{(-x)^j}{(j+\alpha)}\right)\\&=-\frac{1}{\alpha}I_0\left(2\sqrt{b}\right)+\sum_{j=1}^{\infty}b^{-j/2}I_j\left(2\sqrt{b}\right)\left(\frac{(-b/x)^j}{(j-\alpha)}-\frac{(-x)^j}{(j+\alpha)}\right)\\&=-\frac{1}{\alpha}I_0\left(2\sqrt{b}\right)+\sum_{j=1}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-\alpha)}-\frac{\left(x/\sqrt{b}\right)^j}{(j+\alpha)}\right)I_j\left(2\sqrt{b}\right).\end{aligned}\tag{16}$$

Here, we used the power series expansion for I_j, the Modified Bessel Function of the First Kind, order j; see [15, (9.6.10)]. Inserting this result (16) into (15) we find the expansion in terms of I_j

$$\Gamma(\alpha,x;b)=\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right)-\frac{x^\alpha}{\alpha}I_0\left(2\sqrt{b}\right)+x^\alpha\sum_{j=1}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-\alpha)}-\frac{\left(x/\sqrt{b}\right)^j}{(j+\alpha)}\right)I_j\left(2\sqrt{b}\right).\tag{17}$$

Another representation, based on the definition of K_α,

$$\begin{aligned}\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right)&=\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}\left[\frac{2}{\pi}\sin(\pi\alpha)K_\alpha\left(2\sqrt{b}\right)+I_\alpha\left(2\sqrt{b}\right)\right]\\&=2b^{\alpha/2}K_\alpha\left(2\sqrt{b}\right)+\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_\alpha\left(2\sqrt{b}\right),\end{aligned}$$

reads

$$\begin{aligned}\Gamma(\alpha,x;b) &= 2b^{\alpha/2}K_\alpha\left(2\sqrt{b}\right)+\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_\alpha\left(2\sqrt{b}\right)-\frac{x^\alpha}{\alpha}I_0\left(2\sqrt{b}\right)\\ &\quad+x^\alpha\sum_{j=1}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-\alpha)}-\frac{\left(x/\sqrt{b}\right)^j}{(j+\alpha)}\right)I_j\left(2\sqrt{b}\right).\end{aligned} \tag{18}$$

Exploring the behaviour of $I_j\left(2\sqrt{b}\right)$ as $b^{j/2}/j!$ for $j\to\infty$ and for fixed argument, see among others [16, (9.37)], it is clear that this sum converges.

This expression cannot be used for integer values of α. Therefore, we have to study the limit behaviour as $\alpha\to 0$, and $\alpha\to n\,(n\geq 1)$. For values of $\alpha=-n\,(n\geq 1)$ we can apply the functional relation (7).

First, we study the behaviour for $\alpha\to 0$. The following limit relations hold

$$\frac{\pi}{\sin(\pi\alpha)}=\frac{1}{\alpha}+O(\alpha),\quad \alpha\to 0, \tag{19}$$

$$I_{-\alpha}\left(2\sqrt{b}\right)=I_0\left(2\sqrt{b}\right)+\alpha K_0\left(2\sqrt{b}\right)+O(\alpha^2),\quad \alpha\to 0, \tag{20}$$

$$b^{\alpha/2}=\mathrm{e}^{(\alpha/2)\ln b}=1+(\alpha/2)\ln b+O\left(\alpha^2\right),\quad \alpha\to 0, \tag{21}$$

$$\frac{x^\alpha}{\alpha}=\frac{\mathrm{e}^{\alpha\ln x}}{\alpha}=\frac{1+\alpha\ln x+O\left(\alpha^2\right)}{\alpha}=\frac{1}{\alpha}+\ln x+O(\alpha),\quad \alpha\to 0. \tag{22}$$

See [15, (9.6.46)] for (20). This gives for $\alpha\to 0$

$$\begin{aligned}\Gamma(0,x;b) &= \lim_{\alpha\to 0}\left[\left(\frac{1}{\alpha}+O(\alpha)\right)\left(1+(\alpha/2)\ln b+O\left(\alpha^2\right)\right)\left(I_0\left(2\sqrt{b}\right)+\alpha K_0\left(2\sqrt{b}\right)+O(\alpha^2)\right)\right.\\ &\quad\left.-\left(\frac{1}{\alpha}+\ln x+O(\alpha)\right)I_0\left(2\sqrt{b}\right)+x^\alpha\sum_{j=1}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-\alpha)}-\frac{\left(x/\sqrt{b}\right)^j}{(j+\alpha)}\right)I_j\left(2\sqrt{b}\right)\right]\\ &= K_0\left(2\sqrt{b}\right)+\ln\left(\sqrt{b}/x\right)I_0\left(2\sqrt{b}\right)+\sum_{j=1}^{\infty}\frac{1}{j}(-1)^j\left(\left(\sqrt{b}/x\right)^j-\left(x/\sqrt{b}\right)^j\right)I_j\left(2\sqrt{b}\right).\end{aligned} \tag{23}$$

This is equivalent with [1, (38)]. Based on (15) and (23) we can find an expression for $\Gamma(0,x;0)$, which should be equal to the Exponential Integral $E_1(x)$,

$$\begin{aligned}\Gamma(0,x;0) &= \lim_{b\to 0}\lim_{\alpha\to 0}\Gamma(\alpha,x;b)\\ &= \lim_{b\to 0}\lim_{\alpha\to 0}\left[\frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_{-\alpha}\left(2\sqrt{b}\right)+x^\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n\right]\\ &= \lim_{b\to 0}\left[\left(K_0\left(2\sqrt{b}\right)+\ln\left(\sqrt{b}/x\right)I_0\left(2\sqrt{b}\right)\right)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n\right]\\ &= \lim_{b\to 0}\left[\left(K_0\left(2\sqrt{b}\right)+\ln\left(\sqrt{b}/x\right)I_0\left(2\sqrt{b}\right)\right)\right]-\sum_{m=1}^{\infty}\frac{1}{m!m}(-x)^m\\ &= -\gamma-\ln(x)-\sum_{m=1}^{\infty}\frac{1}{m!m}(-x)^m\\ &= E_1(x).\end{aligned} \tag{24}$$

The last limit is true since

$$\lim_{b\to 0}\left(K_0\left(2\sqrt{b}\right)+\ln\left(\sqrt{b}\right)I_0\left(2\sqrt{b}\right)\right)=-\gamma, \tag{25}$$

in [15, (9.6.13)] and the last line is given in [15, (5.1.11)].

Second, we study the behaviour as $\alpha \to n\,(n \geq 1)$ of the general result (17). For $\alpha = n+\delta$, $\delta \to 0$, the following limiting relations hold

$$\frac{\pi}{\sin(\pi\alpha)} = \frac{(-1)^n}{\delta} + O(\delta), \quad \delta = \alpha - n \to 0, \tag{26}$$

$$I_\alpha\left(2\sqrt{b}\right) = I_n\left(2\sqrt{b}\right) + \delta R + O(\delta^2), \quad \delta \to 0, \tag{27}$$

$$\text{with } R = (-1)^n\left[-K_n\left(2\sqrt{b}\right) + \frac{n!b^{-n/2}}{2}\sum_{k=0}^{n-1}(-1)^k\frac{b^{k/2}I_k\left(2\sqrt{b}\right)}{(n-k)k!}\right]. \tag{28}$$

See [15, (9.6.44)] for (27). This gives for $\alpha \to n$, using the second formulation (18)

$$\begin{aligned}
\Gamma(n,x;b) &= \lim_{\alpha\to n}\left[2b^{\alpha/2}K_\alpha\left(2\sqrt{b}\right) + \frac{\pi}{\sin(\pi\alpha)}b^{\alpha/2}I_\alpha\left(2\sqrt{b}\right) - \frac{x^\alpha}{\alpha}I_0\left(2\sqrt{b}\right)\right.\\
&\quad\left. + x^\alpha\sum_{j=1}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-\alpha)} - \frac{\left(x/\sqrt{b}\right)^j}{(j+\alpha)}\right)I_j\left(2\sqrt{b}\right)\right]\\
&= 2b^{n/2}K_n\left(2\sqrt{b}\right)\\
&\quad + \lim_{\delta\to 0}\left[\left(\frac{(-1)^n}{\delta} + O(\delta)\right)b^{n/2}\left(1 + (\delta/2)\ln b + O\left(\delta^2\right)\right)\left(I_n\left(2\sqrt{b}\right) + \delta R + O(\delta^2)\right)\right.\\
&\quad - \frac{x^{n+\delta}}{(n-\delta)}I_0\left(2\sqrt{b}\right) + x^n\left(1 + \delta\ln x + O\left(\delta^2\right)\right)(-1)^n\left(\frac{\left(\sqrt{b}/x\right)^n}{(-\delta)} - \frac{\left(x/\sqrt{b}\right)^n}{(2n+\delta)}\right)I_n\left(2\sqrt{b}\right)\\
&\quad\left. + x^{n+\delta}\sum_{j=1,j\neq n}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-(n+\delta))} - \frac{\left(x/\sqrt{b}\right)^j}{(j+n+\delta)}\right)I_j\left(2\sqrt{b}\right)\right].
\end{aligned} \tag{29}$$

Evaluation of the limit and insertion of the expression for R (see (28)) Eq. (29) give finally

$$\begin{aligned}
\Gamma(n,x;b) &= b^{n/2}K_n\left(2\sqrt{b}\right) + (-1)^n b^{n/2}\ln\left(\sqrt{b}/x\right)I_n\left(2\sqrt{b}\right) + \frac{n!}{2}\sum_{k=0}^{n-1}(-1)^k\frac{b^{k/2}I_k\left(2\sqrt{b}\right)}{(n-k)k!}\\
&\quad - \frac{x^n}{n}I_0\left(2\sqrt{b}\right) - (-1)^n x^n\frac{\left(x/\sqrt{b}\right)^n}{(2n)}I_n\left(2\sqrt{b}\right) + x^n\sum_{j=1,j\neq n}^{\infty}(-1)^j\left(\frac{\left(\sqrt{b}/x\right)^j}{(j-n)} - \frac{\left(x/\sqrt{b}\right)^j}{(j+n)}\right)I_j\left(2\sqrt{b}\right).
\end{aligned} \tag{30}$$

4. Application

A well-known result for $\Gamma(\alpha,x;b)$ with $\alpha = 1/2$ reads

$$\Gamma\left(\frac{1}{2},x;b\right) = \frac{\sqrt{\pi}}{2}\left(\exp\left(2\sqrt{b}\right)\operatorname{erfc}\left(\sqrt{x}+\sqrt{b/x}\right) + \exp\left(-2\sqrt{b}\right)\operatorname{erfc}\left(\sqrt{x}-\sqrt{b/x}\right)\right); \tag{31}$$

see among others [4], [10, (2.120)]. The result (18) gives for $x = \sqrt{b}$

$$\begin{aligned}
\Gamma\left(\frac{1}{2},\sqrt{b};b\right) &= \pi b^{1/4}I_{-1/2}\left(2\sqrt{b}\right) - 2b^{1/4}I_0\left(2\sqrt{b}\right) + b^{1/4}\sum_{j=1}^{\infty}\frac{4(-1)^j}{4j^2-1}I_j\left(2\sqrt{b}\right)\\
&= b^{1/4}\left[\frac{\sqrt{\pi}\cosh\left(2\sqrt{b}\right)}{b^{1/4}} - 2I_0\left(2\sqrt{b}\right) + \sum_{j=1}^{\infty}\frac{4(-1)^j}{4j^2-1}I_j\left(2\sqrt{b}\right)\right],
\end{aligned} \tag{32}$$

so

$$\Gamma\left(\frac{1}{2},\sqrt{b};b\right) = \frac{\sqrt{\pi}}{2}\left(\exp\left(2\sqrt{b}\right)\mathrm{erfc}\left(2b^{1/4}\right)+\exp\left(-2\sqrt{b}\right)\right)$$
$$= b^{1/4}\left[\frac{\sqrt{\pi}\cosh\left(2\sqrt{b}\right)}{b^{1/4}} - 2I_0\left(2\sqrt{b}\right) + \sum_{j=1}^{\infty}\frac{4(-1)^j}{4j^2-1}I_j\left(2\sqrt{b}\right)\right],$$

which is equivalent with

$$\mathrm{erfc}\left(2b^{1/4}\right) = 1 + \frac{2}{\sqrt{\pi}}\exp\left(-2\sqrt{b}\right)b^{1/4}\left[-2I_0\left(2\sqrt{b}\right) + \sum_{j=1}^{\infty}\frac{4(-1)^j}{4j^2-1}I_j\left(2\sqrt{b}\right)\right],$$

or

$$\mathrm{erfc}\,(y) = 1 - \frac{y}{\sqrt{\pi}}\exp\left(-y^2/2\right)\left[2I_0\left(y^2/2\right) + \sum_{j=1}^{\infty}\frac{4(-1)^{j+1}}{4j^2-1}I_j\left(y^2/2\right)\right]. \tag{33}$$

This is equivalent to a form of Neumann's Expansion (see [15, (9.1.82)]) for the function (with $i=\sqrt{-1}$)

$$\frac{\sqrt{\pi}}{2}\exp(-iw)\frac{\mathrm{erf}\left(\sqrt{-2iw}\right)}{\sqrt{-2iw}} = J_0(w) - 2\sum_{j=1}^{\infty}\frac{i^j}{4j^2-1}J_j(w). \tag{34}$$

Moreover, (33) is equivalent with the formula given in [17, Section 4.8.2., (1), p. 122] as

$$\mathrm{erf}\,(y) = \frac{2y}{\sqrt{\pi}}\exp\left(-y^2/2\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\left[I_j\left(y^2/2\right) + I_{j+1}\left(y^2/2\right)\right], \tag{35}$$

by regrouping the terms in the sum.

5. Numerical approach

We have been able to present the Generalized Incomplete Gamma Function $\Gamma(\alpha, x; b)$ in terms of K_α and a series of Bessel functions, I_j. This representation can be evaluated quite easily, in particular for values $x/\sqrt{b}$ close to unity. For large values of $x/\sqrt{b}$ or $\sqrt{b}/x$ convergence of (17), (18) and (23) or (30) is not fast, because of the cancellation of large terms with opposite signs.

Here, we concentrate on the evaluation of the double sum in formula (15) without the factor x^α

$$M = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n,$$

in a Matlab script. We exclude the elements $m = n$, since those elements sum up explicitly to $-\frac{1}{\alpha}I_0\left(2\sqrt{b}\right)$; see (16). Moreover, we exclude the terms for which $n-m-l=0$, where $l=\lfloor\alpha\rfloor\geq 1$, where $\alpha=\lfloor\alpha\rfloor+\delta$, with $0<\delta<1$. Those terms sum up to $-(-1)^l\frac{(\sqrt{b}/x)^l}{\delta}I_l\left(2\sqrt{b}\right)$ for $\delta>0$ or can be handled explicitly for $\delta=0$; see (30). So,

$$M = M' - \frac{1}{\alpha}I_0\left(2\sqrt{b}\right), \quad l=0,$$

or

$$M = M' - \frac{1}{\alpha}I_0\left(2\sqrt{b}\right) - (-1)^l\frac{\left(\sqrt{b}/x\right)^l}{\delta}I_l\left(2\sqrt{b}\right), \quad l\geq 1,$$

where

$$M' = \sum_{m=0,n-m\neq 0,n-m-l\neq 0}^{\infty}\ \sum_{n=0}^{\infty}\frac{1}{(n-m-\alpha)m!n!}(-x)^m\left(-\frac{b}{x}\right)^n.$$

The double sum will be approximated by the construction of an $N\times N$-matrix G. If one can make an error estimation for $|E| = x^\alpha\left|M'-G\right|$, one can determine the corresponding value for N such that $|E| < E_{\mathrm{abs}}$, where E_{abs} is some required absolute error criterion. Summation over all elements of this matrix G gives then a fast approximation for M' with an absolute error $|E| < E_{\mathrm{abs}}$. The sum of the terms not taken into account by the matrix G can be represented by

$$\begin{aligned} E &= x^{\alpha}\left(B + B' + C + C'\right), \quad \text{where} \qquad (36)\\ B &= \sum_{m=0}^{N-1}\frac{(-x)^m}{m!}\sum_{n=N,n-m\neq 0,n-m-l\neq 0}^{\infty}\frac{(-b/x)^n}{(n-m-\alpha)n!},\\ B' &= \sum_{m=N}^{\infty}\frac{(-x)^m}{m!}\sum_{n=m+1,n-m\neq 0,n-m-l\neq 0}^{\infty}\frac{(-b/x)^n}{(n-m-\alpha)n!},\\ C &= \sum_{n=0,n-m\neq 0,n-m-l\neq 0}^{N-1}\frac{(-b/x)^n}{n!}\sum_{m=N}^{\infty}\frac{(-x)^m}{(n-m-\alpha)m!},\\ C' &= \sum_{n=N,n-m\neq 0,n-m-l\neq 0}^{\infty}\frac{(-b/x)^n}{n!}\sum_{m=n+1}^{\infty}\frac{(-x)^m}{(n-m-\alpha)m!}. \end{aligned}$$

The sum B represents the terms for $0 \le m \le N-1$, $n \ge N$; the sum B' the terms $n \ge m+1$, $n \ge N$ and $m \ge N$, and analogously for C and C'. So, E represents all terms not summed. We estimate the various terms in a rather crude, but effective way. We delete the restrictions $(n-m\neq 0, n-m-l\neq 0)$ in the following sum expressions.

$$\begin{aligned} |B| &\le \sum_{m=0}^{N-1}\left|\frac{(-x)^m}{m!}\sum_{n=N}^{\infty}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right|\\ &\le \sum_{m=0}^{N-1}\left|\frac{(-x)^m}{m!}\right|\left|\sum_{n=N}^{N+l-1}\frac{(-b/x)^n}{(n-m-l-\delta)n!} + \sum_{n=N+l}^{\infty}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right|\\ &\le \sum_{m=0}^{N-1}\left|\frac{(-x)^m}{m!}\right|\left\{\left|\sum_{n=N}^{N+l-1}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right| + \frac{(b/x)^{N+l}}{(N+l-m-l-\delta)\,(N+l)!}\right\}\\ &\le \sum_{m=0}^{N-1}\left|\frac{(-x)^m}{m!}\right|\left\{\frac{l\,(b/x)^N}{(1-\delta)N!} + \frac{(b/x)^{N+l}}{(N-(N-1)-\delta)\,(N+l)!}\right\}\\ &\le \sum_{m=0}^{N-1}\frac{x^m}{m!}\left\{\frac{l\,(b/x)^N}{(1-\delta)N!} + \frac{(b/x)^{N+l}}{(1-\delta)\,(N+l)!}\right\}. \end{aligned}$$

$$\begin{aligned} \left|B'\right| &\le \left|\sum_{m=N}^{\infty}\frac{(-x)^m}{m!}\sum_{n=m+1}^{\infty}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right|\\ &\le \sum_{m=N}^{\infty}\left|\frac{(-x)^m}{m!}\right|\left|\sum_{n=m+1}^{m+l-1}\frac{(-b/x)^n}{(n-m-l-\delta)n!} + \sum_{n=m+l+1}^{\infty}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right|\\ &\le \sum_{m=N}^{\infty}\frac{x^m}{m!}\left(\left|\sum_{n=m+1}^{m+l-1}\frac{(-b/x)^n}{(n-m-l-\delta)n!}\right| + \frac{(b/x)^{m+l+1}}{(m+l+1-m-l-\delta)(m+l+1)!}\right)\\ &\le \sum_{m=N}^{\infty}\frac{x^m}{m!}\left(\frac{\max((l-1),0)\,(b/x)^{m+1}}{(1-\delta)\,(m+1)!} + \frac{(b/x)^{m+l+1}}{(1-\delta)(m+l+1)!}\right). \end{aligned}$$

We required the additional condition $b/(x(N+1)) < 1$, (which can be fulfilled for N large enough) to estimate the sums over n by their first terms, since the terms in these sums are alternating in sign and are monotonously decreasing under that condition. The term $(n-m-l-\delta)$ in the denominator in the finite sum can be estimated by $(1-\delta)$ by the requirement that $n-m-l\neq 0$. This explains also that the upper limit of this sum is $m+l-1$ (and not $m+l$). This results in

$$\begin{aligned} \left|B+B'\right| &\le |B| + \left|B'\right|\\ &\le \sum_{m=0}^{N-1}\frac{x^m}{m!}\left\{\frac{l\,(b/x)^N}{(1-\delta)N!} + \frac{(b/x)^{N+l}}{(1-\delta)\,(N+l)!}\right\}\\ &\quad + \sum_{m=N}^{\infty}\frac{x^m}{m!}\left\{\frac{\max((l-1),0)\,(b/x)^{m+1}}{(1-\delta)\,(m+1)!} + \frac{(b/x)^{m+l+1}}{(1-\delta)(m+l+1)!}\right\}. \qquad (37) \end{aligned}$$

Since $b/(x(N+1)) < 1$, this Eq. (37) can be further estimated as

$$\begin{aligned}\left|B+B'\right| &\le \sum_{m=0}^{N-1}\frac{x^m}{m!}\left(\frac{l\,(b/x)^N}{(1-\delta)N!}+\frac{(b/x)^{N+l}}{(1-\delta)\,(N+l)!}\right)+\sum_{m=N}^{\infty}\frac{x^m}{m!}\left(\frac{\max((l-1),0)\,(b/x)^N}{(1-\delta)N!}+\frac{(b/x)^{N+l}}{(1-\delta)(N+l)!}\right)\\ &\le \mathrm{e}^x\frac{(b/x)^N}{(1-\delta)N!}\left(l+\frac{(b/x)^l}{(N+l)!/N!}\right).\end{aligned} \tag{38}$$

Analogously, under the additional condition $x/(N+1) < 1$, we find

$$\begin{aligned}|C| &= \left|\sum_{n=0}^{N-1}\frac{(-b/x)^n}{n!}\sum_{m=N}^{\infty}\frac{(-x)^m}{(n-m-l-\delta)m!}\right|\\ &\le \left|\sum_{n=0}^{N-1}\frac{(-b/x)^n}{n!}\right|\left|\sum_{m=N}^{\infty}\frac{(-x)^m}{(n-m-l-\delta)m!}\right|\\ &\le \sum_{n=0}^{N-1}\frac{(b/x)^n}{n!}\frac{x^N}{|(n-N-l-\delta)|\,N!}\\ &= \sum_{n=0}^{N-1}\frac{(b/x)^n}{n!}\frac{x^N}{(1+l+\delta)N!}.\\ \left|C'\right| &= \left|\sum_{n=N}^{\infty}\frac{(-b/x)^n}{n!}\sum_{m=n+1}^{\infty}\frac{(-x)^m}{(n-m-l-\delta)m!}\right|\\ &\le \left|\sum_{n=N}^{\infty}\frac{(-b/x)^n}{n!}\right|\left|\sum_{m=n+1}^{\infty}\frac{(-x)^m}{(n-m-l-\delta)m!}\right|\\ &\le \sum_{n=N}^{\infty}\frac{(b/x)^n}{n!}\frac{x^{n+1}}{|(n-(n+1)-l-\delta)|\,(n+1)!}\\ &= \sum_{n=N}^{\infty}\frac{(b/x)^n}{n!}\frac{x^{n+1}}{(1+l+\delta)\,(n+1)!}.\\ \left|C+C'\right| &\le \sum_{n=0}^{N-1}\frac{(b/x)^n}{n!}\frac{x^N}{(1+l+\delta)N!}+\sum_{n=N}^{\infty}\frac{(b/x)^n}{n!}\frac{x^{n+1}}{(1+l+\delta)\,(n+1)!}\\ &\le \sum_{n=0}^{N-1}\frac{(b/x)^n}{n!}\frac{x^N}{(1+l+\delta)N!}+\sum_{n=N}^{\infty}\frac{(b/x)^n}{n!}\frac{x^N}{(1+l+\delta)N!}\\ &= \mathrm{e}^{b/x}\frac{x^N}{(1+l+\delta)N!}.\end{aligned} \tag{39}$$

So, we find as an absolute error estimate the rather simple expression

$$|E| \le x^\alpha\left\{\mathrm{e}^x\frac{(b/x)^N}{(1-\delta)N!}\left(l+\frac{(b/x)^l}{(N+1)_l}\right)+\mathrm{e}^{b/x}\frac{x^N}{(1+l+\delta)N!}\right\},\quad \text{for } b/(x(N+1)) < 1 \quad\text{and}\quad x/(N+1) < 1. \tag{40}$$

For given x and b the parameter N can be taken as large as necessary to fulfil the estimate

$$|E| \le x^\alpha\left\{\mathrm{e}^x\frac{(b/x)^N}{(1-\delta)N!}\left(l+\frac{(b/x)^l}{(N+1)_l}\right)+\mathrm{e}^{b/x}\frac{x^N}{(1+l+\delta)N!}\right\} \le E_{\mathrm{abs}},\quad \text{with } E_{\mathrm{abs}} \text{ by the user required absolute error.} \tag{41}$$

So, for given E_{abs} a value for N can be found, and the matrix G can be constructed and summed.

We performed some numerical experiments using the following four different methods.

I. A reference calculation using numerical integration by means of the Matlab®-code `quadgk`, with a relative precision of 1.E–13 and an absolute precision of 1.E–25.

II. A calculation based on formula (18) and (23) or (30) with a relative precision of 1.E–6 and an absolute precision of 1.E–25.

III. A calculation based on (15), where the size $(N \times N)$ of the matrix G has been determined with $E_{\mathrm{abs}} = 1.\mathrm{E}{-}6$; see (41) for the whole range of the x-vector and b-vector.

IV. Another alternative method based on the representation either with Exponential Integrals for integer α, either with Incomplete Gamma functions for non-integer α, with a relative precision of 1.E–6 and an absolute precision of 1.E–25. See also [4, (2.1)]. The description of this method is as follows.

If α is an integer, $\alpha = m \le 0$:

$$\Gamma(m, x; b) = \sum_{n=0}^{\infty} \frac{(-b/x)^n}{n!} x^m E_{-m+n+1}(x), \quad x \ge \sqrt{b}, \tag{42}$$

and

$$\Gamma(m, x; b) = 2b^{m/2} K_m\left(2\sqrt{b}\right) - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} x^m E_{m+n+1}(b/x), \quad 0 \le x < \sqrt{b}. \tag{43}$$

Here, the definition for E_n, the Exponential Integral, reads

$$E_n(x) = \int_1^{\infty} t^{-n} \exp(-xt)\mathrm{d}t, \quad x > 0,\ n = 0, 1, 2, \ldots. \tag{44}$$

In cases where the index of the Exponential Integral E_{m+n+1}, $m \le 0$, is non-positive, that function has to be interpreted as the function α_m; see [15, (5.1.5)]

$$E_{-m}(x) = \alpha_m(x) = \int_1^{\infty} t^m \exp(-xt)\mathrm{d}t; \quad \text{with } \alpha_0(x) = E_0(x) = \exp(-x)/x.$$

The function $E_m(x)$, $m \ge 2$, can be found by a recursion, starting from $E_1(x)$; the function $\alpha_m(x)$, $m \ge 1$, by a recursion starting from $\alpha_0(x)$.

If α is an integer, $\alpha = m > 0$:

$$\Gamma(m, x; b) = 2b^{m/2} K_m\left(2\sqrt{b}\right) - b^m \Gamma(-m, b/x; b). \tag{45}$$

If α is not an integer, $\alpha > 0$ (see (3)):

$$\Gamma(\alpha, x; b) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \Gamma(\alpha - n, x), \quad x \ge \sqrt{b}, \tag{46}$$

and

$$\Gamma(\alpha, x; b) = 2b^{\alpha/2} K_\alpha\left(2\sqrt{b}\right) - \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} b^\alpha \Gamma(-\alpha - n, b/x), \quad 0 \le x < \sqrt{b}. \tag{47}$$

See (4) for the definition of the Incomplete Gamma function $\Gamma(\alpha, x)$. In cases where the first argument of the Incomplete Gamma function $\Gamma(-m, b/x; b)$, $\Gamma(\alpha - n, x)$ or $\Gamma(-\alpha - n, b/x)$, $n \ge 0$, is non-positive, that function can be found by a backward recurrence relation

$$\Gamma(\alpha - 1, x) = \left(\Gamma(\alpha, x) - x^{\alpha-1} \exp(-x)\right) / (\alpha - 1), \tag{48}$$

up to a value with positive first argument.

If α is not an integer, $\alpha < 0$:

$$\Gamma(\alpha, x; b) = 2b^{\alpha/2} K_\alpha\left(2\sqrt{b}\right) - b^\alpha \Gamma(-\alpha, b/x; b). \tag{49}$$

For the three methods (II, III, and IV) we constructed Matlab scripts where x and b are allowed to be vectors. For the matrix G method (III) the size of the matrix is such that the error requirement E_{abs} is satisfied for all elements of x and b to speed up the calculations and to make profit of the vector structure of Matlab. However, if the ranges of x and/or b are large, it is more likely that some combination of x and b induces a large value of N for the size of the matrix to satisfy the error requirement. In such cases it is better to split up the vectors x and b in smaller pieces.

First, we verified formulae (5) and (6) up to high precision. Identification of (5) with (6) and (18) with (30) gives rise to interesting identities between Bessel functions.

We performed some experiments for 11 values for α ($\alpha = -2, -1.5, -1, -0.5, 0, 0.25, 0, 5, 0.75, 1, 1.5, 2$), 5 decades of each 4 values for x ($x = 10^{-4+i}[1, 2.5, 5, 7.5]$), $i = 0, \ldots, 4$ and 5 decades of each 4 values for b ($b = 10^{-5+j}[1, 2.5, 5, 7.5]$), $j = 0, \ldots, 4$. In about 30% of the $11 \times 5 \times 5$ decades the calculation time for Method III was faster than for Method I.

In book [10] tables have been published for the quotient $\Gamma(\alpha, x; b)/\Gamma(\alpha, 0; b)$ for $\alpha = -3 : 0.5 : -2$, and $-1.75 : 0.25 : 1.75$, and $2 : 0.5 : 3$, for $x = 0.01 : 0.01 : 0.1$, and $0.15 : 0.05 : 1$, and $1.5 : 0.5 : 10$, and for $b = 0, 0.5, 1.0, 1.5, 2.0$, for $\alpha \ge 0$, and $b = 0.5, 1.0, 1.5, 2.0, 2.5$, for $\alpha < 0$. For $\alpha = 0$, $b = 0$, $E_1(x)$ is listed and for $\alpha > 0$, $b = 0$ the quotient $\Gamma(\alpha, x)/\Gamma(\alpha)$. These tables have been recalculated up to the precision stated above with the same vectors x and b. All our four methods agreed with each other, and in general Method III was faster than Methods II and IV. The optimized Matlab-code `quadgk` (Method I) is still the fastest. For some combinations of x and b, cancellation of digits occurred at Method II. In particular, for $x = 0.01$ and $b = 2.5$ and parameter values close by. We noted some discrepancies with [10, Tables 2.1–2.21]. It turned out that these calculations have been done in Fortran with Single Precision (S.M. Zubair, *pers. comm.*). This explains the discrepancy since the last digit specified in these tables is not reliable that way.

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